Floating Point

CSC3501 Computer Organization & Design

Instructors:

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Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2?

$8 \; 0 \; 2 \; 1 \quad \frac{1}{2} \quad \frac{1}{8}$

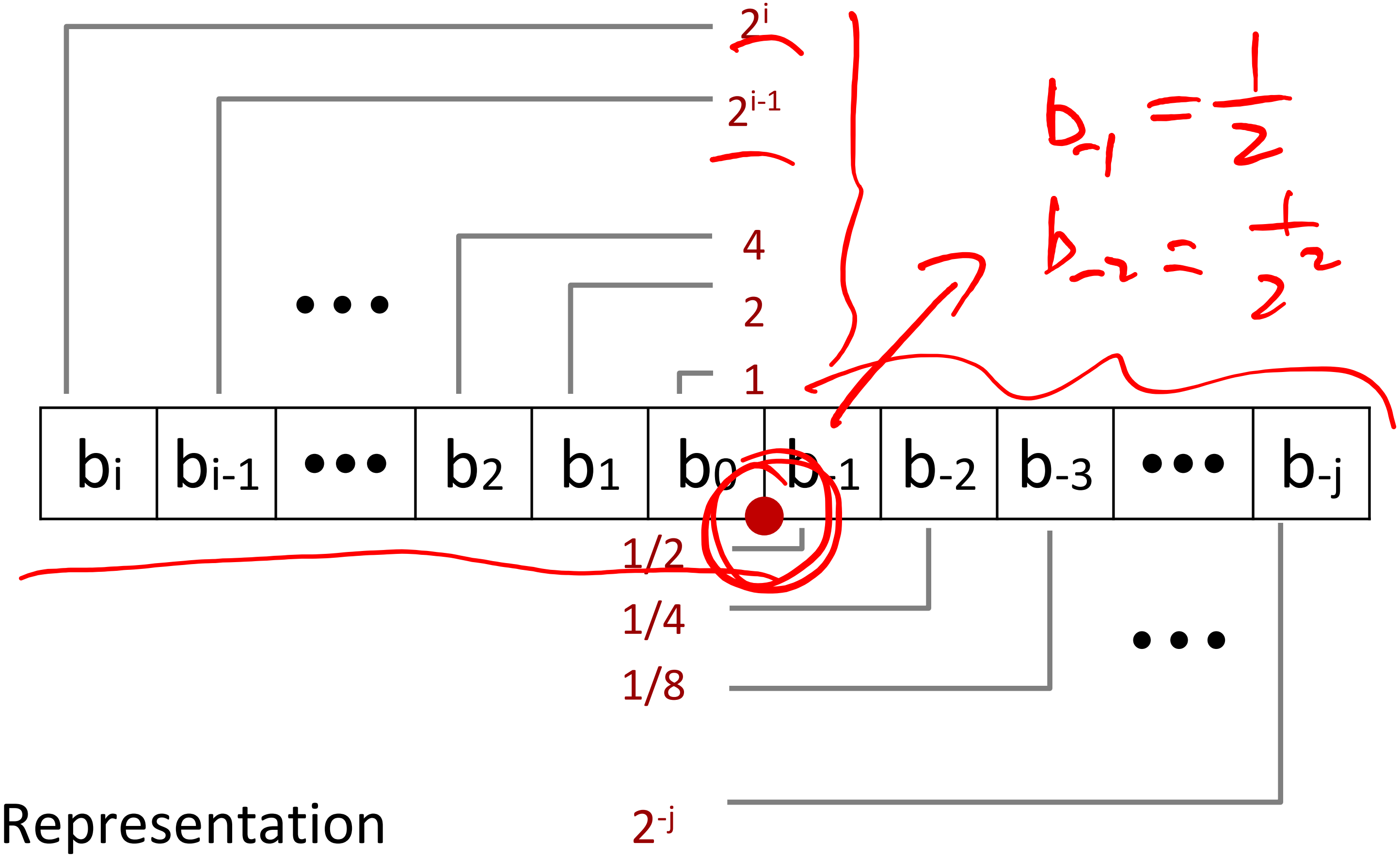
1011.101_2

decimal

$\left(11 + \frac{5}{8}\right)_{10}$

$(11.625)_{10}$

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

- Value / Representation

Value	Representation
5 3/4	5 `101.11`$_2$
2 7/8	`10.111`$_2$
1 7/16	`1.0111`$_2$

$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} \longrightarrow 5 + \frac{3}{4}$

$. \; 2^{-1} \quad 2^{-2} \; . \; . \; . \; .$

- Observations
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form 0.111111…$_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation #1
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations
  - Value   Representation
    - 1/3   `0.0101010101[01]`…2
    - 1/5   `0.001100110011[0011]`…2
    - 1/10   `0.0001100110011[0011]`…2

bits   bits

- Limitation #2
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers (very small values? very large?)

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IEEE 802.x

WiFi

IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

2.5×10^{2} E

- Numerical Form:

$$(-1)^s \, M \; 2^E$$

M

  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0).
  - Exponent E weights value by power of two

- Encoding
  - MSB s is sign bit s
  - exp field encodes E (but is not equal to E)
  - frac field encodes M (but is not equal to M)

s	exp	frac

0 1 1 0 . . . 1 0 . . . 1 0 0

Precision options

- Single precision: 32 bits

s	exp	frac
1	8-bits	23-bits

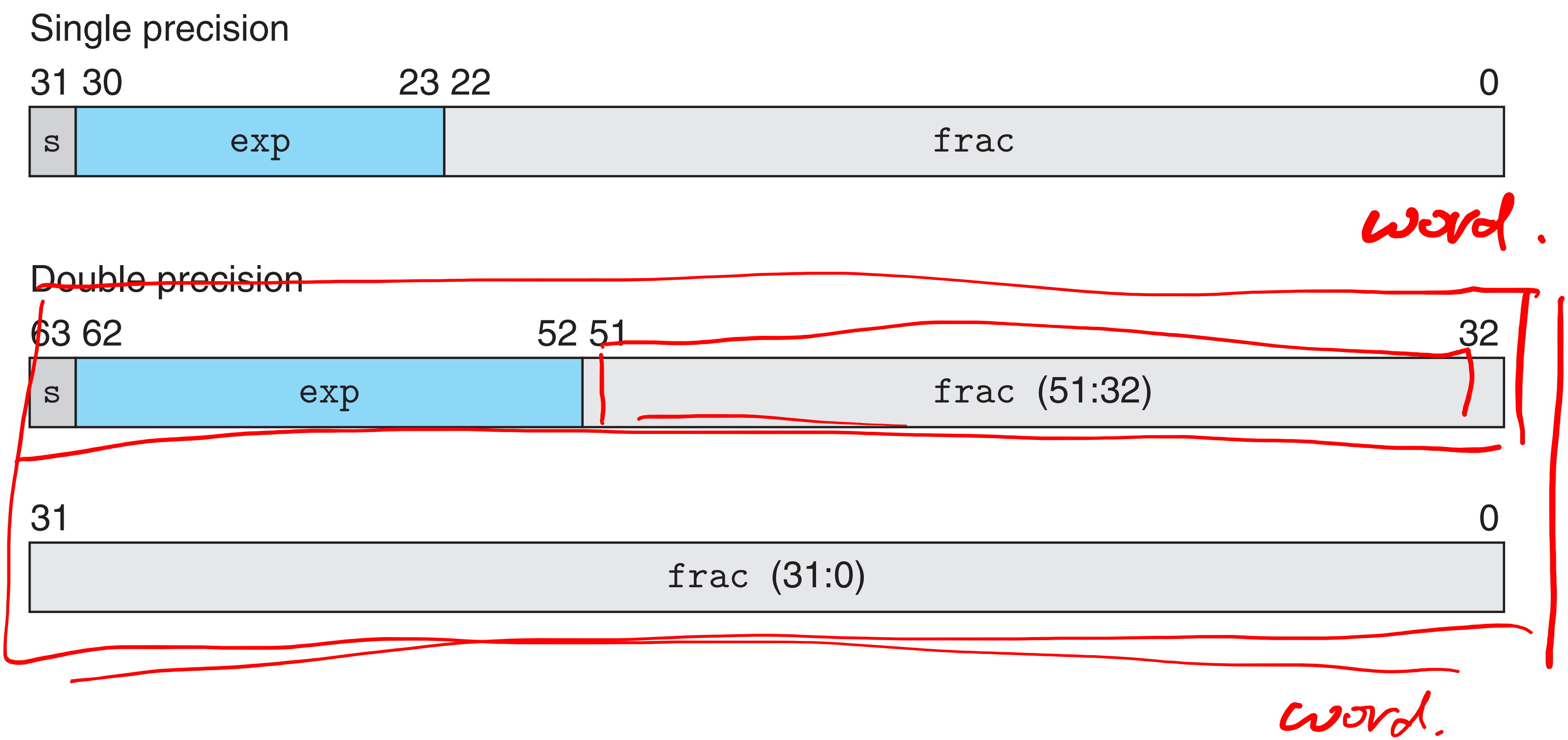
- Double precision: 64 bits

s	exp	frac
1	11-bits	52-bits

- Extended precision: 80 bits (Intel only)

s	exp	frac
1	15-bits	63 or 64-bits

Aligned Memory View



"Normalized" Values

$v = (-1)^s M 2^E$

3bits 4bits

s exp frac

- When: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as a biased value: E = Exp – Bias $= 2^{3-1} - 1 = 3$
  - Exp: **unsigned** value of exp field
  - Bias = 2^{k-1} - 1, where k is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

0 1 1

$E = (011)_2 - $ ___

$(3)_2$

$= 0$

$M = 1 + 0.(\ldots\ldots)$ frac

- Significand coded with *implied leading 1*: M = 1.xxx…x$_2$
  - xxx…x: bits of frac field
  - Minimum when frac=000…0 (M = 1.0)
  - Maximum when frac=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

s 011 exp

Normalized Encoding Example

$v = (-1)^s \, M \, 2^E$
$E = Exp - Bias$

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

13 bits

left shifting

$M = 1 + frac$

- **Significand**

  $M = 1.1101101101101_2$

  $frac = 11011011011010000000000_2$

- **Exponent**

  $E = 13$

  $Bias = 127$

  $Exp = 140 = 10001100_2$

$E = 13$

$E = exp - bias$

$(2^{8-1} - 1) = 127$

- **Result:**

0	10001100	11011011011010000000000
s	exp	frac

$exp = E + Bias\ (127)$
$= 13 + 127 = 140$

exp

frac

2^E

s

$\underbrace{\_\ \_\ \_\ \_}_{4}\ \_\ \_\ \_$

$E = exp - bias$

$2^{4-1} - 1$

7

0000 0

1

1111 15 [-7 ~ 8]

8421

$(-1)^s M \times 2^E$

$M = 1 + frac$

0.1111.

$(-1)^s M \times 2^7 \sim 2^8$

[1, 2)

$2 \sim 2 \times 2^{15}$

Denormalized Values

$v = (-1)^s M\ 2^E$
$E = 1 - \text{Bias}$

- Condition: exp = 000…0
- Exponent value: E = 1 – Bias (instead of E = 0 – Bias)
- Significand coded with implied leading 0: $M = 0.xxx…x_2$
  - **`xxx…x`**: bits of **`frac`**
- Cases
  - **`exp`** = 000…0, **`frac`** = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - **`exp`** = 000…0, **`frac`** ≠ 000…0
    - Numbers closest to 0.0
    - Equispaced

Special Values

- Condition: **`exp`** = **`111…1`**

- Case: **`exp`** = **`111…1`**, **`frac`** = **`000…0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- Case: **`exp`** = **`111…1`**, **`frac`** ≠ **`000…0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

The Three Cases

1. Normalized

s	≠ 0 and ≠ 255	*f*

2. Denormalized

s	0	0	0	0	0	0	0	0	*f*

3a. Infinity

s	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

3b. NaN

s	1	1	1	1	1	1	1	1	≠ 0

Visualization: Floating Point Encodings